NPS-67Gn77061

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ON THE STABILITY OF TWO BASIC PARALLEL FLOWS:
ANALYSIS, PROGRESS REPORT AND PROPOSAL
FOR FURTHER RESEARCH

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JUNE 1977

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS-67Gn77061	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE STABILITY OF TWO BASIC PARALLEL FLOWS: ANALYSIS, PROGRESS REPORT AND PROPOSAL FOR FURTHER RESEARCH		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) T. H. GAWAIN		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 1 June 1977 13. NUMBER OF PAGES 61
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hydrodynamic Stability
Plane Poiseuille Flow
Pipe Poiseuille Flow

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report provides a detailed technical outline and evaluation of recent research performed at the Naval Postgraduate School on the hydrodynamic stability of plane Poiseuille flow and of pipe Poiseuille flow. These two cases are of fundamental importance in connection with the theory of hydrodynamic stability. Each case involves significant discrepancies between the predictions of conventional theory and the results of actual experimental observations. In particular, the conventional theory fails completely to

account for the well known instability of ordinary pipe flow.

This report describes a more general theory which shows promise of overcoming the above limitations. The new theory involves a number of innovations in the formulation of some of the basic equations, in the formu-lation of certain boundary conditions, and in the formulation of the criterion of stability itself. The analytical work is essentially complete and preliminary calculations, while still of limited scope, appear to support the theory and are therefore quite encouraging.

Nevertheless, before the new theory can be considered as definitively confirmed, a considerable amount of systematic computer calculation will still be necessary. Research support is now being sought for this purpose.

SUMMARY

This report provides a detailed technical outline and evaluation of recent research performed at the Naval Postgraduate School on the hydrodynamic stability of plane Poiseuille flow and of pipe Poiseuille flow. These two cases are of fundamental importance in connection with the theory of hydrodynamic stability. Each case involves significant discrepancies between the predictions of conventional theory and the results of actual experimental observations. In particular, the conventional theory fails completely to account for the well known instability of ordinary pipe flow.

This report describes a more general theory which shows promise of overcoming the above limitations. The new theory involves a number of innovations in the formulation of some of the basic equations, in the formulation of certain boundary conditions, and in the formulation of the criterion of stability itself. The analytical work is essentially complete and preliminary calculations, while still of limited scope, appear to support the theory and are therefore quite encouraging.

Nevertheless, before the new theory can be considered as definitively confirmed, a considerable amount of systematic computer calculation will still be necessary. Research support is now being sought for this purpose.

CONTENTS

1. Introduction

The basic research discussed in this report deals with the classical problem of the hydrodynamic stability of parallel flows. Two fundamental cases, both of considerable historical and practical importance, are of particular interest, namely, the plane Poiseuille flow between infinite parallel plates and the flow through a uniform pipe. Despite the considerable amount of research that has been done over the years on these two fundamental cases, the theory available up to now does not account adequately for the observed experimental facts. Indeed, for the case of pipe flow, the classical theory fails to predict the onset of hydrodynamic instability at any Reynolds number whatever! In this respect the existing theory is clearly inadequate.

A critical review of the situation by the present investigator has disclosed that the above limitations and defects of the classical theory can largely be overcome by generalizing the analysis to include certain three dimensional effects that are neglected in the classical approach. The resulting calculations have succeeded for the first time in revealing instabilities in pipe flow for certain types of three dimensional perturbations; these preliminary results appear to be consistent, at least qualitatively, with the known experimental facts about pipe flow. The corresponding calculations for plane Poiseuille flow reveal that under certain conditions instability can occur at a Reynolds number significantly lower than that predicted by the classical theory. The new analysis therefore promises to advance our basic understanding of the important phenomena of hydrodynamic instability in a significant way.

The exploration of the quantitative implications of this generalized three dimensional theory entails a heavy burden of computer calculations. Research support is needed for the purpose of carrying out systematic calculations of this kind. Support is also needed for the purpose of documenting and publishing in the technical literature the analytical basis of the theory and such preliminary numerical results as have already been obtained.

More specifically, it is proposed to carry out the above documentation and publication, as well as a significant amount of the needed additional calculations during the first year of the proposed research. If warranted by the results accomplished during the first year, additional support for follow on research will probably be sought also for another year or two thereafter, to consolidate the progress made.

2. Analytical Approach

The essentials of the analytical approach involved can best be explained by first considering the specific case of plane Poiseuille flow.

All equations are nondimensionalized by utilizing as dimensional reference parameters the density ρ of the fluid, the semi-height a between the walls and the mean volumetric velocity $\bar{U}$ of the fluid.

Cartesian coordinates x, y, z are used with x in the direction of the mean flow and y normal to the walls. Dimensionless coordinate y varies from -1 at the lower wall to $+1$ at the upper wall. Symbols $\vec{i}, \vec{j}, \vec{k}$ denote unit vectors in the x, y, z directions, respectively.

It is well known that the dimensionless velocity U of the mean flow along the x axis is given by the simple expression

$$U = \frac{3}{2}(1 - y^2) \tag{2.1}$$

The hydrodynamic stability of this flow field can be analyzed in the usual way by superimposing upon the above mean flow a suitable perturbation velocity $\vec{v}$ of small amplitude. It is of course necessary that $\vec{v}$ satisfy the continuity equation for incompressible flow, namely,

$$\nabla \cdot \vec{v} = 0 \tag{2.2}$$

A convenient way to ensure that Eq. (2.2) is satisfied is to express $\vec{v}$ in terms of a corresponding vector potential function, let us call it $\vec{W}$, by a relation of the form

$$\vec{v} = \nabla x \vec{W} \tag{2.3}$$

It is customary in linear stability analysis to represent quantities like $\vec{v}$ and $\vec{W}$ by complex variables with the understanding that only

the real part of the variable shall be identified with the actual physical effect in question. This convention simplifies the analytical form of the equations involved.

For a given perturbation vector field $\vec{v}$, the corresponding vector potential function $\vec{W}$ is not uniquely determined but may be expressed in a number of alternative forms all of which are ultimately equivalent. In the present case, however, it proves advantageous to represent the complex vector potential function $\vec{W}$ as follows

$$\vec{W} = [\vec{i}(o) + \vec{j}\, G(y) + \vec{k}\, H(y)]\, e^{(\alpha x + \beta z + \gamma t)} \tag{2.4}$$

where the complex constants α, β, γ may be further expanded to the form

$$\begin{aligned} \alpha &= \alpha_R + i\, \alpha_I \\ \beta &= \beta_R + i\, \beta_I \\ \gamma &= \gamma_R + i\, \gamma_I \end{aligned} \tag{2.5}$$

and where i denotes the imaginary unit, that is,

$$i = + \sqrt{-1} \tag{2.6}$$

The quantities $G(y)$ and $H(y)$ in Eq. (2.4) amount to complex stream functions. Although a single stream function suffices to represent any incompressible plane flow, two stream functions are required to represent an arbitrary incompressible three dimensional flow with complete generality.

Incidentally, this use of two stream functions to represent the perturbation flow in this problem is an innovation introduced by the present investigator. As far as he knows, this method does not appear in the

existing technical literature which deals with the present problem.

Of course, the actual forms of the complex functions G(y) and H(y) are initially unknown but are ultimately calculable in the course of the final numerical solution.

The notation used for the exponential coefficients in Eqs. (2.4) and (2.5) was adopted for reasons of overall generality, consistency and simplicity in the subsequent analysis. Various alternative but equivalent notations are also found in the technical literature, but all such alternatives can be readily translated to the above format whenever necessary.

Parameters α_I and β_I in Eq. (2.5) are associated with the imaginary unit i, and hence they represent the oscillatory wave numbers of the perturbations with respect to space coordinates x and z, respectively. On the other hand α_R and β_R do not involve i and hence they denote real exponential growth rates (or, if negative, decay rates) with respect to space coordinates x and z. Likewise, γ_I represents the oscillatory wave number of the perturbation with respect to time t, that is, the angular frequency, while γ_R represents the exponential growth or decay rate with respect to time. Incidentally, it is convenient to restrict α_I and β_I to non-negative values and there is no loss of generality involved in doing this so long as α_R and β_R remain free to take on positive or negative values as required.

Notice that the above quantities and equations are defined initially in relation to axes that remain fixed with respect to the walls. Later we shall also consider axes that translate at constant velocity relative to the fixed walls. Such motion does not affect parameters α_R, α_I, β_R, β_I but can in general change the apparent values of γ_R and γ_I. We denote these changed values with respect to the moving axes by symbols γ_R' and γ_I'.

In the method of analysis considered here, the numerical values of α_R, α_I, β_R, β_I are assigned arbitrarily, but the corresponding values of γ_R and γ_I must be found by calculation. This point will be clarified further in the subsequent discussion.

Of course it is essential that the resultant perturbed flow field defined by the foregoing relations satisfy Newton's second law of motion. This in turn requires that the velocity and vorticity components satisfy the vorticity transport equation, which is the relation that is found by applying the curl operator to the vector equation of motion. This relation involves terms which are linear in the perturbation quantities and terms which are quadratic in the perturbation quantities. For small perturbations, however, the quadratic terms may be neglected. In this way the customary linearized equation is obtained. A full explanation of these details lies outside the scope of the present discussion, but is given by Harrison, Ref. (6). Fortunately, the result is well known. One convenient version of this result may be summarized concisely as follows

$$\frac{1}{R_e} \nabla x (\nabla x \vec{\omega}) + \nabla x [\vec{\Omega} \times \vec{v} - \vec{U} \times \vec{\omega}] + \left(\frac{\partial \vec{\omega}}{\partial t}\right) = 0 \tag{2.7}$$

where the symbols are defined as indicated below.

$\vec{U} = \vec{i}\,\frac{3}{2}\,(1-y^2)$	Velocity of mean flow field	
$\vec{\Omega} = \nabla x \vec{U}$	Vorticity of mean flow field	
$\vec{v} = \nabla x \vec{W}$	Perturbation velocity	(2.8)
$\vec{\omega} = \nabla x \vec{v}$	Perturbation vorticity	
$R_e =$	Reynolds number	

Notice that Eq. (2.7), the vorticity transport relation, introduces yet another important parameter into the analysis, namely, the Reynolds number R_e .

Of course the solution of Eq. (2.7) must be such as to satisfy the boundary conditions of the problem which are simply that

$$\vec{v} = 0 \quad \text{at} \quad y = \pm 1 \tag{2.9}$$

By a lengthy but systematic procedure which is fully documented in Ref (6), Eq. (2.7) can be expanded in terms of $\vec{W}$ and hence in terms of the functions $G(y)$ and $H(y)$. Eq. (2.7) is a vector relation and therefore implies three corresponding scalar equations. It can be shown, however, that only two of these are independent. Consequently, the final problem reduces to two differential equations in the two unknown functions $G(y)$ and $H(y)$.

In this connection, the following three auxiliary functions occur. These functions are immediately known as soon as definite values are assigned to parameter R_e and to the complex constanst α and β . It is helpful to calculate these functions first, before proceeding with the rest of the solution. Thus

$$\begin{aligned} T(y) &= \frac{(\alpha^2 + \beta^2)}{R_e} - \alpha \frac{3}{2} (1-y^2) \\ I(y) &= \frac{(\alpha^2 + \beta^2)}{R_e} + T(y) \\ J(y) &= (\alpha^2 + \beta^2) \, T(y) - 3\alpha \end{aligned} \tag{2.10}$$

Using symbol D to denote the differential operator $\frac{d}{dy}$, we find that the resulting vorticity transport equation and its associated boundary

condition can now be summarized concisely in the form

$$\left\{\frac{1}{R_e} D^4H + I(y)\, D^2H + J(y)\, H\right\} - \gamma \left\{D^2H + (\alpha^2 + \beta^2)\, H\right\} = 0$$

$$\left.\begin{array}{ll}\text{where} & H = 0 \\ \text{and} & DH = 0\end{array}\right\} \text{ at } y = \pm 1 \tag{2.11}$$

The second vorticity transport equation and its associated boundary condition can likewise be written

$$\left\{\frac{1}{R_e} D^2G + [T(y) - \gamma]\, G\right\} = \frac{\beta}{(\alpha^2 + \beta^2)} \left\{\frac{1}{R_e} D^3H + [T(y) - \gamma]\, DH - 3\alpha yH\right\} \tag{2.12}$$

where $G = 0$ at $y = \pm 1$

Now consider the situation in which the values of parameters α_R, α_I, β_R, β_I and R_e are arbitrarily assigned. Then the three auxilary functions defined by Eqs. (2.10) become known functions. Hence the basic relations given in Eqs. (2.11) can now be solved numerically, subject to the boundary conditions shown.

Eqs. (2.11) define an eigenvalue problem. Theoretically there exist an infinite number of distinct solutions to this equation, each characterize by a corresponding value of the complex constant γ . We term these roots eigenvalues and denote them by symbols γ_1, γ_2, γ_3, ... Associated with each root is a corresponding complex eigenfunction $H_1(y)$, $H_2(y)$, $H_3(y)$... . Fortunately, only the solution which corresponds to the least stable root is required for the stability analysis. The least stable root is defined in more detail later.

Once the kth root γ_k and the kth eigenfunction $H_k(y)$ are known, Eqs. (2.12) may be solved for the corresponding complementary eigenfunction $G_k(y)$, subject to the boundary condition shown.

Notice that Eqs. (2.11) and (2.12) for H and G, respectively, are uncoupled and may therefore be solved sequentially. This fact simplifies the solution. Usually we are interested only in finding the least stable root. In that case it is unnecessary to solve Eqs. (2.12).

By utilizing the method of finite differences, it is possible to reduce the foregoing equations to matrix format. The resulting matrix represents a standard eigenvalue problem which can be solved numerically by familiar and well established techniques.

The details of this reduction to matrix format are presented in Ref. (6) and are not repeated here.

The real part of each eigenvalue is denoted by γ_R and the imaginary part by γ_I. For reasons that are more fully explained later, the eigenvalue that has the algebraically highest value of γ_R is termed the least stable root. We denote this highest real value by symbol γ_{RL}. Of course γ_{RL} depends on the particular numerical values assigned to parameters α_R, α_I, β_R, β_I and R_e. We may symbolize this relation in the form

$$\gamma_{RL} = \gamma_{RL}\,[\alpha_R,\ \alpha_I,\ \beta_R,\ \beta_I,\ R_e] \tag{2.13}$$

The stability of the flow field is determined by the numerical value of γ_{RL} according to the criteria considered in the next two sections.

3. Conventional Criterion of Stability

Historically, the analysis of stability has concentrated attention chiefly on the important but nevertheless special case in which the spatial form of the pertubations is taken to be purely sinusoidal. This amounts to restricting the analysis to the particular case for which, in the present notation,

$$\alpha_R^2 + \beta_R^2 = 0 \tag{3.1}$$

Moreover, any time that Eq. (3.1) applies, the well known theorem of Squire, Ref. (1), also applies. When expressed in terms of the present nomenclature, Squire's theorem simply asserts that for specified values of parameters R_e and α_I, the most unstable condition will always be the one for which

$$\beta_I = 0 \tag{3.2}$$

Thus of the five parameters R_e, α_R, α_I, β_R, β_I which occur in the general case, the constraints expressed by Eqs. (3.1) and (3.2) eliminate three. Therefore the conventional analysis finally retains only two parameters, namely, R_e and α_I.

In the conventional nomenclature the quantity we have called α_I is usually represented by the unsubscripted symbol α ; to avoid ambiguity in the present more general discussion, however, it is important for clarity to adhere consistently to the subscripted notation as introduced originally in connection with Eqs. (2.5).

Under conditions where Eq. (3.1) applies, that is, for the conventional analysis, the stability of the flow field is determined by a very simple rule. The flow is stable, or at least neutrally stable, provided that

$$\gamma_{RL} \leq 0 \tag{3.3}$$

If the inequality applies, the perturbation amplitude decays exponentially over time and the flow is stable. If the equality applies, the perturbation amplitude neither grows nor decays overtime and the flow is neutrally stable. If neither of the above conditions is satisfied, the perturbation amplitude grows with time and the flow is unstable.

Numerical calculations based on the conventional restriction of Eq. (3.1), on the application of Squire's theorem as expressed by Eq. (3.2), and on the simple stability rule summarized in Eq. (3.3), permit the flow stability to be determined for arbitrarily assigned values of R_e and α_I. The region of instability in the R_e vs α_I plane has the general form sketched qualitatively in Fig. 3.1. Notice that there exists a critical Reynolds number below which sinusoidal streamwise perturbations of all wave numbers α_I are stable. The conventional analysis gives the value of the critical Reynolds number, based on channel semi-height and volumetric mean velocity as about 3850, Thomas, Ref. (2). Of course, all of these conventional results are familiar; they have been known for some time.

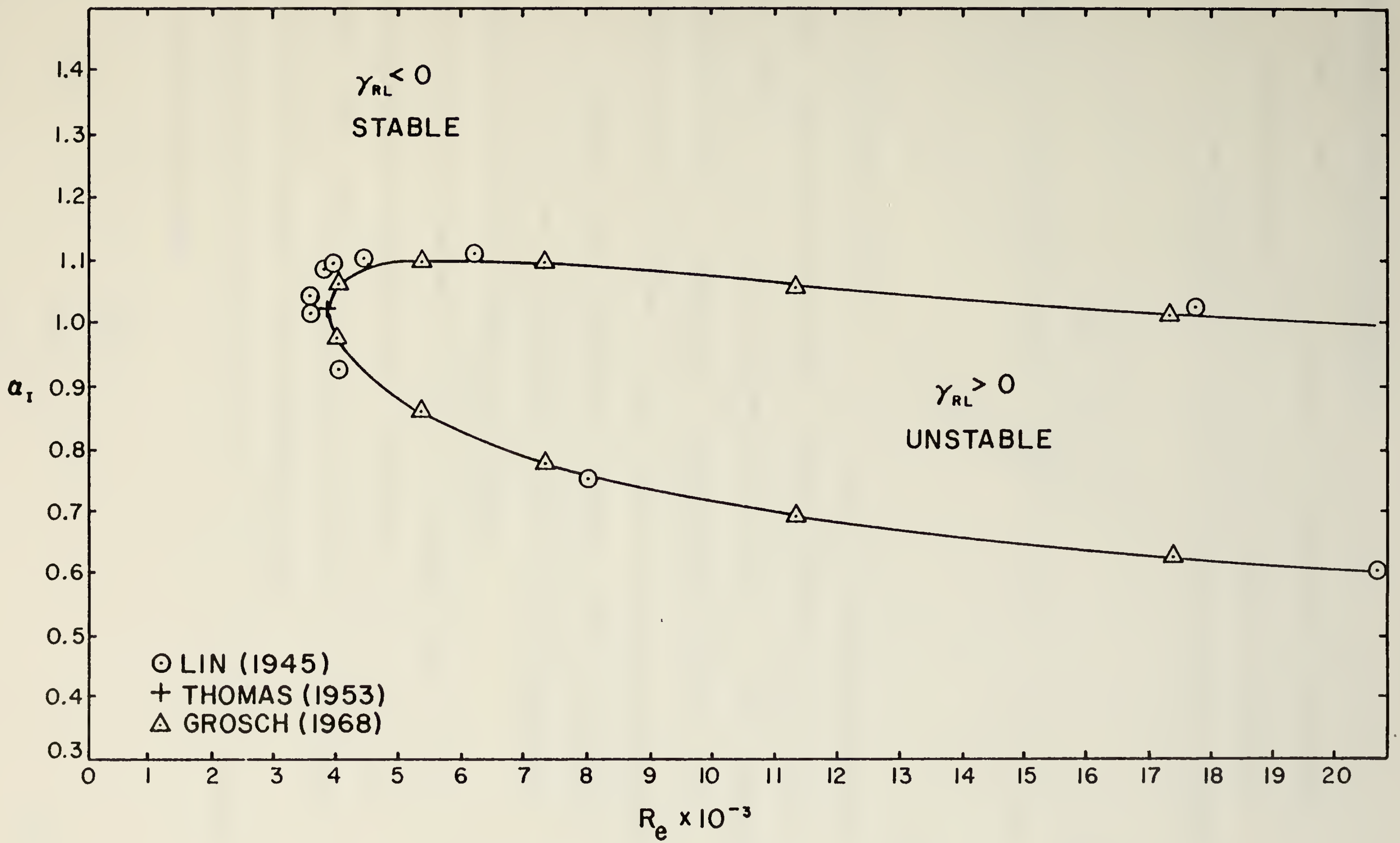


FIG. 3.1 STABILITY OF PLANE POISEUILLE FLOW

4. Generalized Criterion of Stability

The simple stability rule summarized in Eq. (3.3) of the previous section, although correct for the special case of purely sinusoidal oscillations, is not adequate for dealing with the more general situation where

$$(\alpha_R^2 + \beta_R^2) \neq 0 \tag{4.1}$$

Moreover, under these conditions, Squire's theorem no longer holds either, so that

$$\beta_I \neq 0 \tag{4.2}$$

Thus of the five parameters R_e, α_R, α_I, β_R, β_I which occur in the general case, none can now be dropped. Compared with the conventional case which involves only the two parameters R_e and α_I , this represents a considerable increase in complexity. The additional parameters which occur in the generalized problem are associated with certain instabilities which are not revealed in the conventional analysis. This is the basic reason why the generalized analysis gives results in better agreement with experiment than does the conventional method.

While the simple criterion of stability expressed by Eq. (3.3) is universally accepted as applying to the conventional case, there is no such unanimity among different investigators concerning the appropriate stability criterion to apply to the generalized problem.

The analysis of the previous section has shown that if we arbitrarily impose the restrictions that

$$\begin{aligned} \alpha_R &= 0 \\ \beta_R &= 0 \\ \gamma_{RL} &= 0 \end{aligned} \tag{4.3}$$

these conditions are sufficient to ensure neutral stability. Eqs. (4.3) are frequently misinterpreted as representing the conditions which are necessary to ensure neutral stability. In fact Eqs. (4.3), while unquestionably sufficient, are not necessary for this purpose. If they were necessary conditions, they would of course rule out of consideration altogether the generalized case which is the very subject of the research under discussion. Nevertheless, this seems to be a fairly common misunderstanding of the situation, a fact which might help explain why the fully general three dimensional case has not received the degree of research attention that it deserves.

A useful clue to a more satisfactory formulation of a generalized stability criterion is provided by the existance of two special cases in which there seems to be general agreement on how the stability should properly be defined, Refs. (4) and (5). These are summarized in Eqs. (4.4) and (4.5) below.

The flow field is stable, or at least neutrally stable,

$$\left.\begin{array}{ll} \text{if} & \alpha_R = 0 \\ \text{when} & \gamma_{RL} \leq 0 \end{array}\right\} \qquad (4.4)$$

$$\left.\begin{array}{ll} \text{or if} & \gamma_R = 0 \\ \text{when} & \alpha_{RL} \leq 0 \end{array}\right\} \qquad (4.5)$$

A significant feature of Eqs. (4.4) and (4.5) is that they both refer explicitly to parameter α_R but not to β_R . Thus stability is defined in terms of the component of spatial growth rate in the streamwise direction but not in terms of the component in the transverse direction (if any).

In analytical work such as considered primarily in this discussion, parameter α_R can be set to zero independently as required by the first of Eqs. (4.4) and β_R and the other parameters can be chosen arbitrarily. The value of γ_{RL} for the least stable root can be found by calculation, and the resulting stability determined from the second of Eqs. (4.4). While this method is favored by some investigators, it fails for the important case where $\alpha_R \neq 0$.

In a hypothetical experiment involving steady oscillations we would have $\gamma_R = 0$ in accordance with the first of Eqs. (4.5). The observed streamwise growth or decay rate would then fix the value of α_{RL} for the least stable root. The second of Eqs. (4.5) would then fix the stability. While this concept can hardly be faulted for the special case to which it applies, unfortunately it fails for the important case where $\gamma_R \neq 0$.

One basic reason for the diversity of stability criteria utilized by different researchers is that in the general case the apparent exponential growth or decay rate of the perturbation with respect to time, as denoted by parameter γ_R' , depends in part on the motion, if any, of the observer's reference frame. To see this consider axes that translate at constant velocity parallel to the fixed walls. Let $\dot{x}_o$ denote the translation velocity component in the downstream or x direction, and let $\dot{z}_o$ denote the velocity component, if any, in the lateral or z direction. Let x', y, z' denote the coordinates of a point fixed with respect to the moving axes, and let x, y, z denote the coordinates of the same point with respect to fixed axes. Then

$$
\begin{aligned}
x &= x' + \dot{x}_o t \\
z &= z' + \dot{z}_o t
\end{aligned}
\tag{4.6}
$$

The exponent in Eq. (2.4) can now be written as follows

$$
\begin{aligned}
X &= \alpha x + \beta z + \gamma t \\
&= \alpha(x' + \dot{x}_o\, t) + \beta(z' + \dot{z}_o\, t) + \gamma t \\
&= \alpha x' + \beta z' + (\gamma + \alpha\dot{x}_o + \beta\dot{z}_o)t \\
&= \alpha x' + \beta z' + \gamma' t
\end{aligned}
\tag{4.7}
$$

where

$$\gamma' = \gamma + \alpha\dot{x}_o + \beta\dot{z}_o \tag{4.8}$$

Breaking this into real and imaginary parts gives

$$
\begin{aligned}
\gamma'_R &= \gamma_R + \alpha_R\,\dot{x}_o + \beta_R\,\dot{z}_o \\
\gamma'_I &= \gamma_I + \alpha_I\,\dot{x}_o + \beta_I\,\dot{z}_o
\end{aligned}
\tag{4.9}
$$

Thus, with respect to a reference frame which moves with any specified values of $\dot{x}_o$ and $\dot{z}_o$, the flow field may be said to be stable, or at least neutrally stable, provided that for the least stable root

$$\gamma'_{RL} = (\gamma_{RL} + \alpha_R\dot{x}_o + \beta_R\dot{z}_o) \leq 0 \tag{4.10}$$

The least stable root is that root which, for fixed values of the other parameters, has the algebraically largest value of γ'_R as defined above.

Incidentally, for purely sinusoidal oscillations, that is, for

$$\alpha_R = \beta_R = 0 \tag{4.11}$$

Eq. (4.10) reduces to

$$\gamma'_{RL} = \gamma_{RL} \leq 0 \tag{4.12}$$

In other words the stability is the same with respect to all uniformly translating reference frames in this case, irrespective of the values of $\dot{x}_o$ and $\dot{z}_o$ involved. Thus the criterion of stability is simple and unambiguous under these circumstances.

When we revert to the general case, however, the apparent stability as expressed by Eq. (4.10) depends on the velocities $\dot{x}_o$, $\dot{z}_o$ of the chosen reference axes. Thus the concept of instability is not completely and uniquely defined until parameters $\dot{x}_o$ and $\dot{z}_o$ are designated in an appropriate way.

In the present context it is reasonable to require that these velocities be specified in such a way that the resulting stability criterion be consistent with Eqs. (4.4) and (4.5) for the two special cases to which these relations apply. Comparison of Eq. (4.10) with Eqs. (4.4) and (4.5) reveals that this will indeed be the case provided that we set

$$\begin{aligned} \dot{z}_o &= 0 \\ \dot{x}_o &> 0 \end{aligned} \tag{4.13}$$

Consequently, Eqs. (4.9) and (4.10) now simplify to the form

$$\begin{aligned} \gamma'_{RL} &= \gamma_{RL} + \alpha_R \dot{x}_o \leq 0 \\ \gamma'_{IL} &= \gamma_{IL} + \alpha_I \dot{x}_o \end{aligned} \tag{4.14}$$

where the magnitude of $\dot{x}_o$ still remains to be defined. In this connection, a method that is followed by some investigators is to choose $\dot{x}_o$ in such a way as to give

$$\gamma'_{IL} = 0 \tag{4.15}$$

The corresponding value of $\dot{x}_o$, which is now termed the phase velocity, is seen to be

$$\dot{x}_o = - \frac{\gamma_{IL}}{\alpha_I} \tag{4.16}$$

and this quantity is normally positive as required. Consequently, with respect to this particular reference frame, the flow is stable, or at least neutrally stable, provided that

$$\gamma_{RL}' = \gamma_{RL} - (\frac{\alpha_R}{\alpha_I})\ \gamma_{IL} \leq 0 \tag{4.17}$$

It is also possible to define a phase velocity which satisfies Eq. (4.15) in a more general manner. In place of setting $\dot{z}_o = 0$, we may require that

$$\dot{x}_o^{\ 2} + \dot{z}_o^{\ 2} = \text{a minimum} \tag{4.18}$$

However, this option contravenes Eq. (4.13) and will therefore not be considered further.

Despite the fact that Eq. (4.17) is the stability criterion preferred by some writers, this investigator does not recommend its use, for the following reason. Note that the restriction imposed by Eq. (4.15) is essentially arbitrary. It is a restriction on the angular frequency and as such has no necessary relation to growth or decay rate. Consequently the stability criterion expressed by Eq. (4.17), while mathematically correct with respect to the axes to which it pertains, is as arbitrary as are those axes themselves.

To eliminate the arbitrary element involved in Eq. (4.17), this investigator advocates a criterion based on adopting a Lagrangian viewpoint. Consider a given streamline of the mean flow located at some specified value

of coordinate y. The fluid particles which lie along this streamline advance downstream at an average velocity U relative to fixed axes. We now stipulate that the flow along this streamline shall be deemed stable, neutrally stable or unstable, respectively, according to whether the particles advancing along that streamline undergo gradually increasing, unchanging or gradually decreasing amplitudes of oscillation. This amounts to letting the references axes move with the mean velocity of the fluid particles along the chosen streamline. Hence we set

$$\begin{aligned} \dot{x}_o &= U(y) = \frac{3}{2}(1-y^2) \\ \dot{z}_o &= 0 \end{aligned} \tag{4.19}$$

Consequently, the flow along the given streamline maybe said to be stable, or at least neutrally stable, provided that for the least stable root

$$\gamma'_{RL} = \gamma_{RL} + \alpha_R\, U(y) \leq 0 \tag{4.20}$$

According to this Lagrangian criterion of stability, the least stable root is that root which, for given values of α_R, $U(y)$ and the other parameters, has the highest algebraic value of γ'_R. This will also be the root which has the highest value of γ_R with respect to the original fixed axes.

It also follows from the foregoing concept that different streamlines exhibit different degrees of instability (except for the special case where $\alpha_R = 0$). Of particular importance is the least stable streamline, that is, the streamline which, for given values of γ_{RL} and α_R, has the algebraically largest value of γ'_{RL}. Eq. (4.20) reveals that if α_R be

positive, then the streamline at mid channel is the least stable, while if α_R be negative, then the streamline at the wall is the least stable. In either case we can state that a condition of incipient instability is reached when

$$\gamma'_{RL} = 0 \tag{4.21}$$

for the least stable streamline.

From the foregoing considerations we may conclude finally that the flow is stable, or at least neutrally stable,

$$\left.\begin{aligned} &\text{when} \quad \alpha_R > 0 \;, \quad \text{if} \quad \gamma_{RL} \leq -\frac{3}{2}\alpha_R \\ &\text{when} \quad \alpha_R \leq 0 \;, \quad \text{if} \quad \gamma_{RL} \leq 0 \end{aligned}\right\} \tag{4.22}$$

Notice that the least stable root γ_{RL} in Eqs. (4.22) is now the ordinary root with respect to stationary axes; the moving axes have served their purpose and are not needed further. Also notice that the second of Eqs. (4.22) is identical to the conventional stability criterion of Eq. (3.3). Thus only if $\alpha_R > 0$ do we need to modify the conventional stability criterion in the manner shown.

The generalized Lagrangian criterion of stability as developed in this section and summarized concisely in Eqs. (4.22), forms an essential element of the proposed research. This generalization of the stability criterion represents an original contribution of the present investigator and, so far as he is aware, does not duplicate the published work of any other researcher.

Moreover, preliminary results calculated by this method and reported by Harrison, Newby, and Johnston, Refs. (6), (7) and (8), appear reasonable and are encouraging. They show, for example, that the introduction of

negative values of α_R is definitely destabilizing and lowers the critical Reynolds number below that which applies for the conventional case, $\alpha_R = 0$. This is a significant result. Moreover, it is a result that could not have been discovered if the investigation had been guided by the overly restrictive criterion of neutral stability as expressed by Eqs. (4.3).

The present theory also shows that for negative values of α_R instability starts at the walls at some critical Reynolds number, and with gradually increasing Reynolds number it spreads progressively inward. This is in agreement with experimental observation, Schlicting, Ref. (3).

Application of the foregoing concepts permits us to assign the four parameters $\alpha_R, \alpha_I, \beta_R, \beta_I$ arbitrarily and to calculate by a process of numerical trial and correction the corresponding Reynolds number for incipient instability. The results disclosed by calculations of this kind for a suitable range of values of the above four parameters can be symbolized in the form

$$R_{e_I} = R_{e_I}\,[\alpha_R, \alpha_I, \beta_R, \beta_I] \qquad (4.23)$$

Of particular interest is the question whether there exists some particular combination of parameters $\alpha_R, \alpha_I, \beta_R, \beta_I$ for which R_{e_I} attains an absolute minimum value, or whether $\alpha_R, \alpha_I, \beta_R$ and β_I , if otherwise unrestricted, may always be so chosen as to reduce R_{e_I} below any pre-assigned limit. This amounts to asking whether, or in what sense, a critical Reynolds number may be said to exist below which the flow field is stable to all three dimensional perturbations. While this problem has been solved for the special case where $\alpha_R = \beta_R = 0$, it remains relatively unexplored for the far more complex generalized case.

5. Transformation of Parameters for Plane Poiseuille Flow

The foregoing development has shown that the linearized stability problem of plane Poiseuille flow can be formulated in terms of the parameters α_R, α_I, β_R, β_I, and R_e . In this section we introduce an alternative but equivalent set of parameters which are here symbolized as A, ϕ, Θ, κ and $R_e{}^*$. This change in parameters yields an important theoretical result and also permits the numerical calculations to be significantly reduced.

The fundamental relations, Eqs. (2.11) and (2.12), which govern the solution for the eigenvalues and eigenfunctions are seen to contain two important complex constants as shown in the following two equations. The left side of each equation expresses each constant in terms of the four original parameters α_R, α_I, β_R, β_I and the right side in terms of the four new parameters A, ϕ, Θ and κ . Thus

$$
\begin{aligned}
(\alpha_R + i\,\alpha_I)^2 + (\beta_R + i\,\beta_I)^2 &= A^2 e^{\,2\phi} \\
(\alpha_R + i\,\alpha_I) &= \kappa A e^{\,(\phi + \Theta)}
\end{aligned}
\qquad (5.1)
$$

Eqs. (5.1) suffice to fix the solution for the four new parameters A, ϕ, Θ and κ as functions of the four original parameters α_R, α_I, β_R, β_I or vice versa. The details of these conversions in either sense are given by Newby, Ref. (7), and are not repeated here. Newby also gives certain auxiliary relations associated with this transformation of parameters which need not be considered in the present discussion.

It is useful, however, to define here the important auxiliary parameter

$$\lambda_R^2 = \alpha_R^2 + \beta_R^2 \tag{5.2}$$

The relationships symbolized by Eqs. (5.1) can also be summarized graphically in the complex plane as shown in Fig. 5.1.

The vector OP in the diagram represents the complex quantity α^2. The vector PQ represents β^2. The geometry of the diagram is completely fixed when the coordinates of the points P and Q are specified. Notice that the locations of these two points suffice to determine the four characteristics α_R, α_I, β_R, β_I or A, ϕ, Θ and κ which fix the spatial wave form of the perturbation.

Consider the Reynolds number which corresponds to incipient instability for a perturbation whose spatial wave form has been specified. This relation is symbolized by Eq. (4.23) which we repeat here for easy reference, namely,

$$R_{eI} = R_{eI}\,[\alpha_R, \alpha_I, \beta_R, \beta_I] \tag{4.23}$$

A remarkable and important feature of the transformation defined by Eqs. (5.1) and (5.2) is that it permits the relation symbolized by Eq. (4.23) to be simplified significantly as explained below.

The transformation in question makes use of a special reference case which, for arbitrary values of parameters A, ϕ and Θ, is characterized by a particular value of the fourth parameter, namely,

$$\kappa \to \kappa^* = 1 \tag{5.3}$$

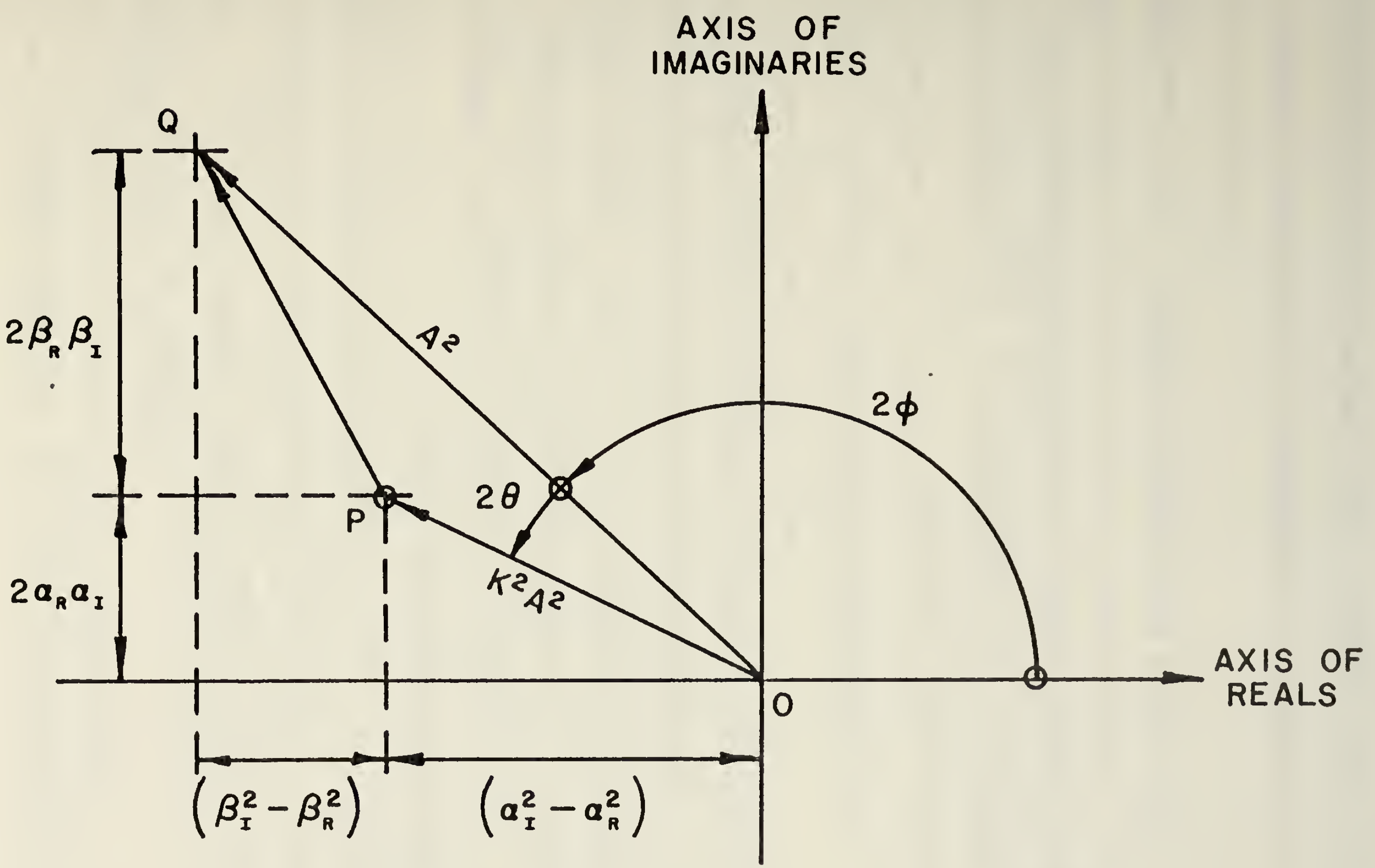


FIG. 5.1 SPATIAL WAVE FORM PARAMETERS

It can also be shown from Eqs. (5.1) that under these conditions the parameter λ_R^2 also takes on a corresponding reference value, namely,

$$\lambda_R^2 \rightarrow \lambda_R^{*2} = A^2 \left[|\sin \Theta| + \cos^2 \phi \right] \tag{5.4}$$

For the above reference case, the solution of the incipient stability problem analogous to that symbolized by Eq. (4.23) reduces to a form which may be symbolized as follows.

$$R_{eI} \rightarrow R_{eI}^* = R_{eI}^* \left[A, \phi, \Theta \right] \tag{5.5}$$

The important feature of Eq. (5.5) as compared with Eq. (4.23) is that the number of independent parameters appearing on the right has been reduced from four to three, namely, to A, ϕ, and Θ. Owing to the length and complexity of the matrix calculations involved, this reduction from four to three independent parameters represents a very great reduction in the overall computational burden.

The problem that now remains, however, is how to generalize from the above special reference case to the more general case in which parameters A, ϕ and Θ remain fixed at their previous values, but in which κ is allowed to vary arbitrarily. Of course parameters λ_R^2 and R_{eI} also vary accordingly. It can be shown, however, that these variations are governed by the following simple analytical relations, namely,

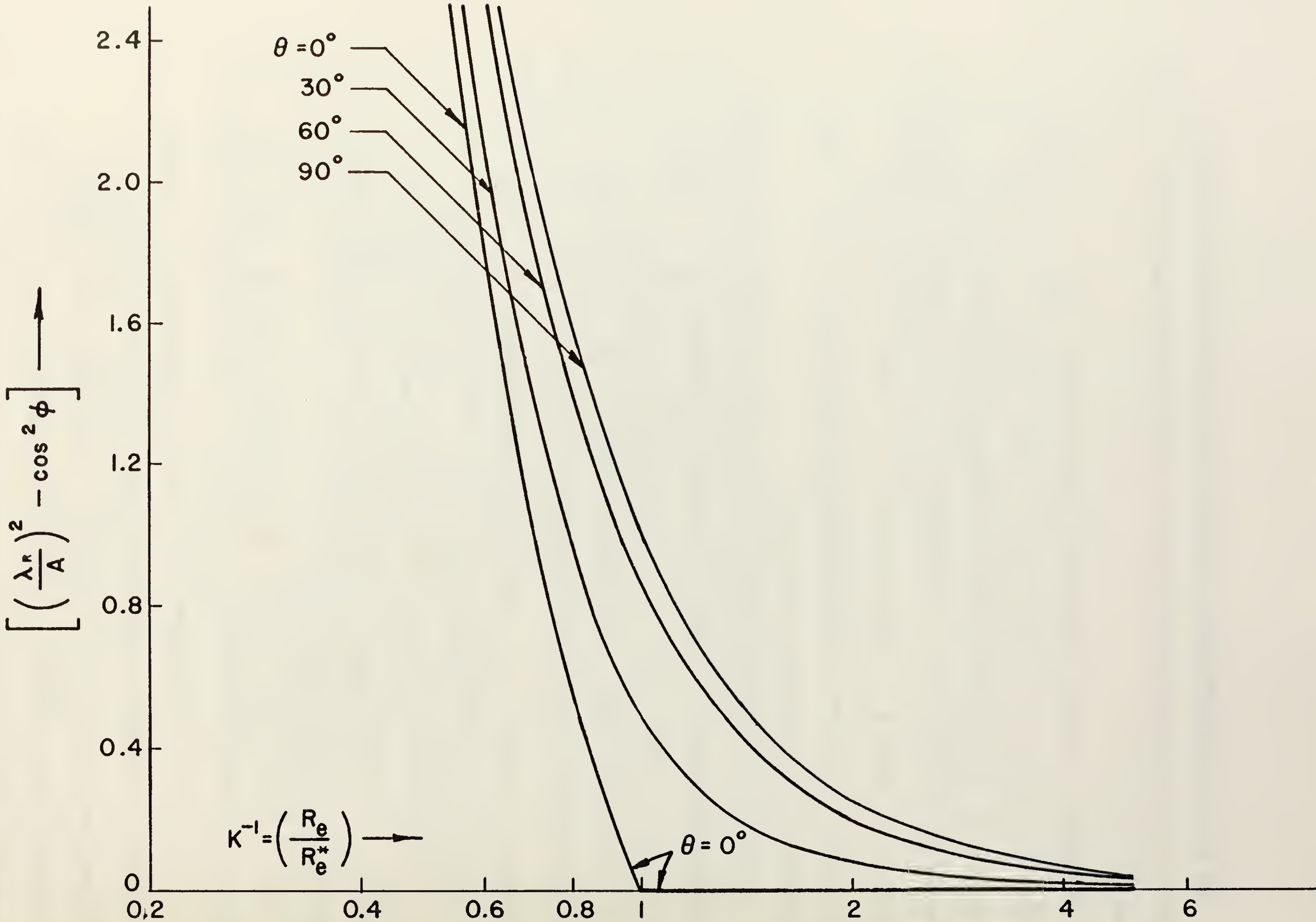
$$\left(\frac{R_{eI}}{R_{eI}^*} \right) = \kappa^{-1} = \sqrt{\frac{\left[\left(\frac{\lambda_R}{A} \right)^2 - \cos^2 \phi + \sin^2 \Theta \right]}{\left[\left(\frac{\lambda_R}{A} \right)^2 - \cos^2 \phi \right] \left[\left(\frac{\lambda_R}{A} \right)^2 + \sin^2 \phi \right]}} \tag{5.6}$$

The detailed derivation of the relations symbolized by Eqs. (5.3) through (5.6) lies outside the scope of this discussion but is fully documented in Ref. (7).

The general nature of the relation defined by Eq. (5.6) is displayed graphically in Fig. 5.2. Notice that for fixed values of A, ϕ and Θ, an increase in λ_R^2 always implies a corresponding decrease in κ^{-1} and hence a decrease in R_{eI}. Moreover, if λ_R^2 be allowed to increase without bound, then R_{eI} approaches zero in the limit. On the other hand if λ_R^2 is limited to some specified upper bound, then R_{eI} cannot drop below some corresponding lower bound.

The foregoing discussion is based on holding parameters A, ϕ and Θ fixed and considering the effects of possible changes in λ_R^2. Let us next consider the case where λ_R^2 is held fixed and parameters A, ϕ and Θ are free to vary. It follows from Eqs. (5.5) and (5.6) that there now exists some definite combination of values for A, ϕ and Θ, respectively, for which R_{eI} attains an absolute minimum value. We term this the critical Reynolds number $R_e)_{cr}$. It is evident that $R_e)_{cr}$ is some definite function of the single remaining independent parameter, λ_R^2. <u>One important and specific objective of the proposed research is to compute this function</u>.

Eqs. (5.5) and (5.6) show that for an assigned value of λ_R^2, the critical combination of values of A, ϕ and Θ which minimize the value of R_{eI} cannot be found analytically, but must be determined by a systematic process of numerical exploration over the four dimensional domain whose coordinates are A, ϕ, Θ and R_e^*. Some preliminary numerical calculation of this kind has already been accomplished on a limited scale, and research support is needed for the purpose of extending this exploration


θ = 0°
30°
60°
90°
θ = 0°
$\left[\left(\frac{\lambda_R}{A}\right)^2 - \cos^2\phi\right]$
$K^{-1} = \left(\frac{R_e}{R_e^*}\right)$
2.4
2.0
1.6
1.2
0.8
0.4
0
0.2
0.4
0.6
0.8
1
2
4
6

in a systematic way. This task forms an essential part of determining the critical Reynolds number as a definite numerical function of λ_R^2.

The theory developed in this section shows that, even before any specific numerical calculations are undertaken, the so-called critical Reynolds number is not some absolute constant as it is commonly conceived to be, but rather some definite function of λ_R^2. This is a significant new concept and represents an original contribution by the present investigator.

For the special case where

$$\lambda_R^2 = 0 \tag{5.7}$$

the present theory and transformations reduce to the conventional and well known relations associated with Squire's theorem. This investigator knows of no published result, however, which is fully equivalent to the generalized form of these transformation as outlined in this section. These results are believed to represent an original contribution by the present investigator.

The analysis of the generalized criterion of stability as presented in the previous section has shown that the conventional constraint expressed by Eq. (5.7), which is widely regarded as a necessary condition for neutral stability, is not in fact necessary at all. The analysis of this section now shows that positive values of λ_R^2 have the further effect of decreasing the critical Reynolds number. This result helps explain why the conventional analysis tends to predict values of critical Reynolds number which are somewhat too high and how slight differences in experimental set-up or test conditions, which might be associated with different values of λ_R^2, could produce apparent inconsistencies in the observed critical Reynolds number. A significant point is that this explanation of discrepancies is still

based on a relatively straightforward linearized analysis whereas some investigators, failing to find any simple explanation of this kind, have felt compelled to resort to non-linear analyses, with all of the enormous complexities they entail, in an effort to make sense out of the experimental observations.

6. Stability of Pipe Flow

While the stability of plane Poiseuille flow as discussed in the previous sections of this report is of fundamental interest because of the basic simplicity of the three dimensional rectilinear geometry involved, this configuration is of somewhat limited practical importance, and it does not lend itself readily to experimental study. In fact the amount of experimental data available for this case is extremely limited.

Pipe flow on the other hand, while involving a geometry which is only slightly more complex, is of the greatest practical importance. Also, there exists an extensive body of experimental data on the stability of pipe flow. Moreover, the discrepancy between the conventional theory and experimental observation is extreme in this case in that the theory fails completely to predict the experimentally observed instability. These are strong reasons for generalizing the theoretical analysis for pipe flow to include three dimensional effects in a manner analogous to that previously discussed in connection with plane Poiseuille flow. Preliminary calculations based on such a generalized analysis, while of very limited scope, have succeeded for the first time in disclosing theoretical instabilities in pipe flow. This is a noteworthy accomplishment and warrants a sustained follow up.

Broadly speaking, the pipe flow problem can be analyzed by much the same method previously explained in connection with plane Poiseuille flow. Only certain details are different in the two cases. An extensive analysis for pipe flow is given by Harrison, Ref. (6) and Johnston, Ref. (8), and is not repeated here. However, we do point out in this section the main respects in which the analysis of the stability of pipe flow differs from that of plane Poiseuille flow.

All equations are nondimensionalized by utilizing as dimensional reference parameters the density ρ of the fluid, the radius a of the pipe and the mean volumetric velocity $\bar{U}$ of the fluid.

Cylindrical coordinates x, r, Θ are used with x being in the direction of the flow, r being in the radial direction and Θ being the angular coordinate. Dimensionless coordinate r varies from zero at the axis to $+1$ at the pipe wall. Symbols $\vec{e}_x, \vec{e}_r, \vec{e}_\theta$ denote unit vectors in the three coordinate directions.

It is well known that the dimensionless velocity U of the mean flow along the x axis is given by the simple expression

$$U = 2(1 - r^2) \tag{6.1}$$

It is of interest to compare this relation for pipe flow with the corresponding relation for plane flow as given by Eq. (2.1). Notice that the maximum dimensionless velocity now equals 2 as compared with 3/2 for plane flow.

The vector potential of the perturbation is chosen in the form

$$\vec{W}_n = [\vec{e}_x(o) + \vec{e}_r G_n(r) + \vec{e}_\Theta H_n(r)] e^{(\alpha x + i n\Theta + \gamma t)} \tag{6.2}$$

where the complex constants α and γ may be expanded further as follows.

$$\begin{aligned} \alpha &= \alpha_R + i \alpha_I \\ \gamma &= \gamma_R + i \gamma_I \end{aligned} \tag{6.3}$$

Notice that the coefficient of θ in the exponent in now taken not as a general complex quantity β, but rather as the purely imaginary quantities $i n$ where n is an integer. Thus

$$\beta = \beta_R + i\,\beta_I = 0 + i\,n$$

$$\beta_R = 0 \tag{6.4}$$

$$\beta_I = n \qquad n = 0, 1, 2, 3, \ldots$$

The reasons for restricting the complex coefficient β in this way is that the velocity components, and hence the vector potential itself, must be strictly periodic with respect to angle Θ . This is a fortunate circumstance since it eliminates parameter β_R and reduces β_I to an integer, thus simplifying the problem. The spatial form of the perturbation is therefore completely defined by the three parameters α_R, α_I and n .

As Refs. (6) and (8) show in full detail, the vorticity transport equations that result from the use of the above vector potential are found to reduce to the following form

$$[M_4'] \begin{Bmatrix} D^4G_n \\ D^4H_n \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3G_n \\ D^3H_n \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2G_n \\ D^2H_n \end{Bmatrix} + [M_1'] \begin{Bmatrix} DG_n \\ DH_n \end{Bmatrix}$$

$$+ [M_o'] \begin{Bmatrix} G_n \\ H_n \end{Bmatrix} = \gamma \left([N_2'] \begin{Bmatrix} D^2G_n \\ D^2H_n \end{Bmatrix} + [N_1'] \begin{Bmatrix} DG_n \\ DH_n \end{Bmatrix} + [N_o'] \begin{Bmatrix} G_n \\ H_n \end{Bmatrix} \right. \tag{6.5}$$

where symbol D now denotes the differential operator d/dr .

The eight matrices $[M']$ and $[N']$ which appear in Eq. (6.5) each have two rows and two columns, or 32 elements in all, of which 10 happen to be zeros. All elements are defined in detail in Refs. (6) and (8). These definitions are too lengthy to be repeated here. It suffices to remark that all of these matrix elements are known complex functions of the three spatial wave form parameters α_R, α_I and n , of Reynolds number R_e ,

and of the radial coordinate r . Moreover, the coefficients of the quantities D^3G_n and D^4G_n in Eqs. (6.5) are zeros, so that these equation are of only second order in G_n whereas they are of fourth order in H_n . Thus only two boundary conditions are needed which involve G_n or its derivatives, while four are needed which involve H_n or its derivatives.

An interesting feature of Eqs. (6.5) is that, in general, these two equations are coupled, that is, both functions G_n and H_n and their various derivatives appear in both equations. Hence the equations must be solved simultaneously. For the special case $n = 0$, however, these equations uncouple. One of them becomes an eigenvalue equation of fourth order in the single unknown function H_o . The other becomes an independent eigenvalue equation of second order in the single unknown function G_o . Th the solution for $n = 0$ breaks down into two independent families of eigen- values and eigenfunctions. It can be shown that one of these families represents perturbations in the meridional plane θ = constant , the other represents perturbations in the cross-sectional plane x = constant . On th other hand for $n = 1, 2, 3 \ldots$ the eigenfunctions G_n and H_n which correspond to a particular eigenvalue remain coupled and therefore represent perturbations of a truly three dimensional character.

Solution of the basic vorticity transport relations, Eqs. (6.5) must be found subject to the appropriate boundary conditions at the wall, $r = 1$ and at the axis, $r = 0$. The boundary conditions are discussed in some detail in the next section of this report.

If numerical values of α_R, α_I , n and R_e be specified, the real part γ_{RL} of the least stable root can be found by solution of the vorticit transport equations. The stability can then be determined by a Lagrangian criterion analogous to Eq. (4.22). In other words the flow is stable, or

at least neutrally stable,

$$\left.\begin{array}{llll} \text{when} & \alpha_R > 0 \ , & \text{if} & \gamma_{RL} \leq -2\alpha_R \\ \text{when} & \alpha_R \leq 0 \ , & \text{if} & \gamma_R \leq 0 \end{array}\right\} \qquad (6.6)$$

Applications of the foregoing concepts permits us to assign the three parameters α_R, α_I and n arbitrarily and to calculate by a process of numerical trial and correction the corresponding Reynolds number for incipient instability. The results disclosed by calculation of this kind for a suitable range of values of the above three parameters can be symbolized in the form

$$R_{eI} = R_{eI}\,[\alpha_R, \alpha_I, n] \qquad (6.7)$$

Of particular interest is the question whether, if we hold α_R constant at some arbitrary value, there exists some particular combination of parameters α_I and n for which R_{eI} attains an absolute minimum value. We term this the critical Reynolds number $R_e)_{cr}$. It is evident that $R_e)_{cr}$ is some definite function of the single remaining independent parameter, α_R . <u>One important and specific objective of the proposed research is to compute this function</u>.

While the stability of pipe flow has been extensively investigated for the special case $\alpha_R = 0$, it nevertheless remains relatively unexplored for the more general case in which $\alpha_R \neq 0$. Moreover, in pipe flow certain boundary conditions at the axis become so problematical as to raise serious questions about the validity of any purported solution that does not properly take these subtleties into account. These aspects are considered in the next two sections.

7. Boundary Conditions for Pipe Flow

In pipe flow, the final vorticity transport equations are of second order in function G_n and of fourth order in function H_n. Hence two boundary conditions are needed with respect to G_n and four with respect to H_n. It turns out that three of these conditions are defined at the wall, $r = 1$, and the remaining three at the axis, $r = 0$.

The boundary conditions at the wall, $r = 1$, are based on the usual no slip restriction. Thus, for all values of index n

$$\left.\begin{aligned} u_n(1) &= 0 \\ v_n(1) &= 0 \\ w_n(1) &= 0 \end{aligned}\right\} \tag{7.1}$$

Inasmuch as the governing differential equations are finally expressed in terms of the two basic functions $G_n(r)$ and $H_n(r)$, it is useful to rewrite Eqs. (7.1) in like terms. In this connection recall that the perturbation velocity $\vec{v}_n$ is related to the vector potential $\vec{W}_n$ as follows.

$$\vec{v}_n = \nabla x \vec{W}_n = \left[\vec{e}_x u_n(r) + \vec{e}_r v_n(r) + \vec{e}_\Theta\, w_n(r)\right] e^{(\alpha x + in\Theta + \gamma t)} \tag{7.2}$$

From Eqs. (6.2) and (7.2) we readily find that the three velocity components are given by the following expressions.

$$\begin{aligned} u_n &= DH_n + \frac{1}{r}(H_n - in\,G_n) \\ v_n &= -\ \alpha H_n \\ w_n &= +\ \alpha G_n \end{aligned} \tag{7.3}$$

It is also useful in the present context to expand functions G_n and H_n in Maclaurin series. Thus

$$\begin{aligned} G_n &= G_n(o) + DG_n(o)\ r + D^2G_n(o)\ \frac{r^2}{2!} + \dots \\ H_n &= H_n(o) + DH_n(o)\ r + D^2H_n(o)\ \frac{r^2}{2!} + \dots \end{aligned} \tag{7.4}$$

With the aid of Eqs. (7.3) and (7.4) it becomes a straightforward task to rewrite Eqs. (7.1) in the required form. The following results are obtained which apply for all values of index n, namely,

$$\begin{aligned} G_n(1) &= 0 \\ H_n(1) &= 0 \\ DH_n(1) &= 0 \end{aligned} \tag{7.5}$$

Derivation of the remaining three boundary conditions which apply at the axis $r = 0$ is considerably more complicated. We start by observing that Eq. (7.2) correctly gives the perturbation velocity $\vec{v}_n$ at any point which is <u>not</u> on the axis but does not necessarily define a meaningful velocity at a point on the axis. To express the velocity on the axis it is not sufficient merely to set $r = 0$ in Eq. (7.3). The reason is that Eq. (7.2) contains the quantities $\vec{e}_r$, $\vec{e}_\Theta$ and $e^{in\Theta}$ which are functions of Θ, whereas coordinate Θ, while determinate for any point off the axis, is indeterminate at the axis itself.

We may circumvent this difficulty by starting at some arbitrary point near but not on the axis, and proceeding toward the axis along a radial line for which Θ remains constant. In the limit as $r \to 0$, the final location of the point on the axis is independent of the coordinate Θ of the line

along which that point is approached. Likewise, the velocity $\vec{v}_n$ at that point is independent of the coordinate Θ which characterizes the path of approach. Hence we may write

$$\frac{\partial}{\partial\Theta}\left\{\lim_{r\to 0} \vec{v}_n (x, r, \Theta, t)\right\} = 0 \tag{7.6}$$

In applying Eq. (7.6) to Eq. (7.2) we make use of the fact that $\vec{e}_x$ is a constant but $\vec{e}_r$ and $\vec{e}_\Theta$ are functions of Θ , so that

$$\begin{aligned} \left(\frac{\partial \vec{e}_x}{\partial\Theta}\right) &= 0 \\ \left(\frac{\partial \vec{e}_r}{\partial\Theta}\right) &= + \vec{e}_\Theta \\ \left(\frac{\partial \vec{e}_\Theta}{\partial\Theta}\right) &= - \vec{e}_r \end{aligned} \tag{7.7}$$

It now follows easily from Eqs. (7.2), (7.6) and (7.7) that for arbitrary Θ and arbitrary n

$$\begin{aligned} n\, u_n(o) &= 0 \\ i\, n\, v_n(o) - w_n(o) &= 0 \\ v_n(o) + in\, w_n(o) &= 0 \end{aligned} \tag{7.8}$$

Eqs. (7.8) stipulate the conditions that must be satisfied in order that Eq. (7.2) shall define a unique and determinate value of $\vec{v}_n$ in the limit as $r \to 0$. The value of Θ at this limit may be chosen arbitrarily without affecting the resulting value of $\vec{v}_n$.

Eqs. (7.8) break down into three distinct cases, namely, $n = 0$, $n = 1$ and $n \geq 2$. Thus

For $n = 0$

$$v_o(o) = 0$$
$$w_o(o) = 0 \tag{7.9}$$

Notice that $u_o(o)$ does not vanish in this case.

For $n = 1$

$$u_1(o) = 0$$
$$v_1(o) + i\, w_1(o) = 0 \tag{7.10}$$

Notice that $v_1(o)$ and $w_1(o)$, although mutually constrained, do not vanish in this case.

For $n = 2, 3, 4, 5, \ldots$

$$u_n(o) = 0$$
$$v_n(o) = 0 \tag{7.11}$$
$$w_n(o) = 0$$

It is significant that $u_o(o)$, $v_1(o)$ and $w_1(o)$ are the only non-zero velocity components at the axis. Thus the cases $n = 0$ and $n = 1$ suffice to account fully for the three non zero perturbation velocity components that can occur at this location.

It is now useful to express Eqs. (7.9), (7.10) and (7.11) in terms of functions G_n and H_n. The following results are obtained

For $n = 0$

$$G_o(o) = 0$$
$$H_o(o) = 0 \tag{7.12}$$

Note that one additional boundary condition is still needed for this case.

For $n = 1$

$$H_1(o) - i\,G_1(o) = 0$$

$$2DH_1(o) - i\,DG_1(o) = 0 \tag{7.13}$$

Note that one additional boundary condition is still needed for this case also.

Notice in addition that this case is exceptional in that it represents the only value of n for which $H_n(o)$ and $G_n(o)$ do not vanish.

For $n = 2, 3, 4, 5, \ldots$

$$G_n(o) = o$$

$$H_n(o) = 0 \tag{7.14}$$

$$2DH_n(o) - i\,n\,DG_n(o) = 0$$

It is now clear that for $n \geq 2$, Eqs. (7.5) and (7.14) fix all six of the boundary conditions needed to define a determinate solution. On the other hand the cases $n = o$ and $n = 1$ each still lack one final necessary boundary condition.

The one remaining boundary condition needed for the case $n = 0$ can readily be deduced from well known physical laws. Consider a small fluid element of cylindrical shape, of length δx and radius δr, whose axis coincides with the x coordinate axis. Let τ_{xr} denote the longitudinal shear stress acting on the curved surface of the element, let p be the pressure, ρ the density and $(\frac{Du}{dt})$ the axial acceleration. We may write the equation of motion in the x direction for this element as follows.

$$\tau_{xr}\, 2\pi\delta r\, \delta x - (\frac{\partial p}{\partial x})\, \delta x \pi\, (\delta r)^2 = \rho\, \pi(\delta r)^2\, \delta x\, (\frac{Du}{dt}) \tag{7.15}$$

Proceeding to the limit $\delta r \to 0$, we observe that the last two terms of Eq. (7.15) are of higher order than the first. Hence

$$\lim_{\delta r \to 0} \tau_{xr} = \lim_{r \to 0} \tau_{xr} = 0 \tag{7.16}$$

It then follows from the known relation between stress and strain rate for a viscous fluid that the corresponding shear strain rate also vanishes along the axis. It can readily be shown that this requires in turn that

$$\lim_{r \to 0} \left\{ Du + \alpha v \right\} = 0 \tag{7.17}$$

Upon expressing this in terms of functions G_o and H_o and upon making use of Eqs. (7.9), we find that the vanishing of the shear strain rate requires finally that

$$D^2 H_o(o) = 0 \tag{7.18}$$

This fixes the last boundary condition needed for the case $n = 0$.

All of the boundary conditions required for the pipe flow problem are therefore determined above except one. This is the last boundary condition needed to define a determinate solution for the important case $n = 1$.

Unfortunately, this last required boundary condition is also the most difficult to formulate in a clear, unequivocal and rigorous manner. The axis represents a singular line along which many of the usual analytical expressions for quantities like $\vec{v}$, $\vec{\omega}$, $\nabla x \vec{\omega}$ and so on break down. The functions G_n and H_n and their various derivatives, especially the higher derivatives, are not necessarily continuous at $r = 0$. Consequently, intuitive notions based on various symmetry relations and

continuity assumptions become unreliable guides.

We call attention at this point to the analogy that exists between perturbation vorticity $\vec{\omega}_n$ and perturbation velocity $\vec{v}$ in this flow field. Thus

$$\vec{\omega}_n = \nabla \times \vec{v}_n = \left[\vec{e}_x \, \xi(r) + \vec{e}_r \, \eta(r) + \vec{e}_\Theta \, \zeta(r)\right] e^{(\alpha x + in\Theta + \gamma t)} \tag{7.19}$$

The analogy between this relation and Eq. (7.2) implies that if Eq. (7.19) is to remain valid at $r = 0$, then we must have, by analogy with Eqs. (7.8),

$$\begin{aligned} n \, \xi_n(o) &= 0 \\ i \, n \, \eta_n(o) - \zeta_n(o) &= 0 \\ \eta_n(o) + in \, \zeta_n(o) &= 0 \end{aligned} \tag{7.20}$$

In particular for $n = 1$, these reduce to

$$\begin{aligned} \xi_1(o) &= 0 \\ \eta_1(o) + i \, \zeta_1(o) &= 0 \end{aligned} \tag{7.21}$$

Of course Eqs. (7.13) already supply two of the three needed boundary conditions, and Eqs. (7.21) are under consideration here only in connection with the one additional constraint that is still lacking. If we invoke the first of Eqs. (7.21) for this purpose, the following constraint is obtained, namely,

$$DH_1(o) - i \, 2DG_1(o) = 0 \tag{7.22}$$

On the other hand, if we invoke the second of Eqs. (7.21) for the same purpose, a quite different result is obtained, namely

$$D^2H_1(o) - i\ D^2G_1(o) = 0 \qquad (7.23)$$

Unfortunately, with only one degree of freedom still at our disposal, there is no way to satisfy both Eq. (7.22) and Eq. (7.23) simultaneously. Hence both of Eqs. (7.21) cannot be satisfied simultaneously either. This simply means that for the case in question, $n = 1$, Eq. (7.19) inevitably breaks down at $r = 0$. Moreover, the attempt to utilize the analogy between Eq. (7.19) and (7.2) leads to an overspecification of the needed final boundary condition.

Further study is needed in order to reformulate the required final boundary condition at the axis in a way that is unambiguous and free of arbitrary assumptions. This is an important question, and the difficulties involved in correctly formulating this particular boundary condition might perhaps account for the failure of the conventional theory to disclose the known instability of pipe flow. A cricial review of this question will form an important part of the proposed research.

For the present, we shall tentatively assume that the first of Eqs. (7.21) provides the appropriate final boundary condition, and that the second of these equations may be discarded. This amounts to retaining Eq. (7.22) and discarding Eq. (7.23).

Upon combining Eqs. (7.13) and (7.22), we find that the three final boundary conditions at the axis simplify for this case to the following form.

For $n = 1$

$$DG_1(o) = 0$$

$$H_1(o) - i\, G_1(o) = 0 \tag{7.24}$$

$$DH_1(o) = 0$$

The final overall boundary conditions for all values of n are now as summarized in Eqs. (7.25) through (7.28) of Table 7.1.

Table 7.1 Boundary Conditions for Pipe Flow

For n = 1

$$G_o(1) = 0 \qquad G_o(o) = 0 \qquad (7.25)$$

$$H_o(1) = 0 \qquad DH_o(1) = 0 \qquad H_o(o) = 0 \qquad D^2H_o(o) = 0 \qquad (7.26)$$

This case is exceptional in that the solutions for functions G_n and H_n become uncoupled for $n = 0$.

For n = 1

$$G_1(1) = 0 \qquad DG_1(o) = 0$$

$$H_1(1) = 0 \qquad H_1(o) - i\, G_1(o) = 0 \qquad (7.27)$$

$$DH_1(1) = 0 \qquad DH_1(o) = 0$$

This case is exceptional in that the quantities $G_n(o)$ and $H_n(o)$ become non-zero for $n = 1$.

For n = 2, 3, 4, 5, ...

$$G_n(1) = 0 \qquad G_n(o) = 0$$

$$H_n(1) = 0 \qquad H_n(o) = 0 \qquad (7.28)$$

$$DH_n(1) = 0 \qquad 2DH_n(o) - in\, DG_n(o) = 0$$

8. Radial Symmetry Relations and Continuity Conditions at Axis

It is quite common in dealing with a variety of physical problems which involve a cylindrical geometry to derive certain boundary conditions at the axis from considerations of symmetry. Hence it is natural to inquire whether any such constraints apply also to the pipe flow problem. The main purpose of the discussion in this section is to show that although the solution of the present problem does indeed satisfy certain relations of radial symmetry, the attempt to derive the boundary conditions on this basis involves certain hidden pitfalls. We may even conjecture that such pitfalls might be a factor in the failure of conventional efforts to find any theoretical instability in pipe flow, although we know from experience that such instability does in fact occur.

With reference to symmetry, consider an arbitrary point P with coordinates x, r, Θ . A diametrically opposite point P' may now be located with coordinates $x, r, (\Theta + \pi)$. While negative values of coordinate r are not ordinarily used, we can elect in this instance to employ them, and to designate the coordinates of point P' in the alternative but equivalent form $x, -r, \Theta$. Of course the perturbation velocity at point P' , call it $\vec{v}'$, must be the same regardless of which of the above two alternative coordinate descriptions are employed. Thus

$$\vec{v}' [x, -r, \Theta, t] = \vec{v}' [x, +r, (\Theta + \pi), t] \qquad (8.1)$$

Harrison, Ref. (6), has shown that as a consequence of the relation symbolized by Eq. (8.1), the functions G_n and H_n must satisfy symmetry constraints of the form

$$G_n(-r) = -(-1)^n G_n(+r)$$
$$H_n(-r) = -(-1)^n H_n(+r) \tag{8.2}$$

This tells us that if n be even, functions G_n and H_n have odd symmetry with respect to r, while if n be odd, G_n and H_n have even symmetry.

Consequently, for n even,

$$G_n(-r) = -G_n(+r)$$
$$D^2G_n(-r) = -D^2G_n(+r) \tag{8.3}$$
$$D^4G_n(-r) = -D^4G_n(+r) \quad \text{etc.}$$

and similarly for function H_n and its even derivatives.

Likewise, for n odd,

$$DG_n(-r) = -DG_n(+r)$$
$$D^3G_n(-r) = -D^3G_n(+r) \quad \text{etc.} \tag{8.4}$$

and similarly for the odd derivatives of function H_n.

If we now assume that the functions and derivatives in question are also continuous at $r = 0$, then the above relations have the following further consequences.

<u>For n even</u>

$$G_n(o) = 0 \qquad H_n(o) = 0$$
$$D^2G_n(o) = 0 \qquad D^2H_n(o) = 0 \tag{8.5}$$
$$D^4G_n(o) = 0 \qquad D^4H_n(o) = 0$$
$$\text{etc.} \qquad \text{etc.}$$

For n odd

$$DG_n(o) = 0 \qquad DH_n(o) = 0$$
$$D^3G_n(o) = 0 \qquad D^3H_n(o) = 0 \tag{8.6}$$
$$\text{etc.} \qquad \text{etc.}$$

An error that is sometimes made is to assume that the known symmetry relations expressed by Eqs. (8.2) necessarily imply the consequences summarized in Eqs. (8.5) and (8.6). In fact, however, these consequences apply if and only if the quantities in question are known to be continuous at $r = 0$. Inasmuch as the mathematical functions ordinarily encountered in connection with incompressible flows are usually highly continuous, it is easy to overlook the fact that Eqs. (8.5) and (8.6) actually involve a hidden assumption of continuity that may or may not be justified in the present instance.

The situation is further clouded by the fact that a number of the boundary conditions derived in the previous section from entirely different considerations, and summarized in Table 7.1, happen to coincide with the symmetry/continuity relations of Eqs. (8.5) and (8.6). These are the following

For $n = 0$

$$G_o(o) = 0$$
$$H_o(o) = 0 \tag{8.7}$$
$$D^2H_o(o) = 0$$

For $n = 1$

$$DG_1(o) = 0$$
$$DH_1(o) = 0 \tag{8.8}$$

For $n = 2, 4, 6, \ldots$

$$G_n(o) = 0$$
$$H_n(o) = 0 \tag{8.9}$$

Moreover, for the case $n = 3, 5, 7, \ldots$, if Eqs. (8.6) could be assumed to apply, they would also ensure that the condition

$$2DH_n(o) - in\, DG_n(o) = 0 \tag{8.10}$$

is satisfied identically.

The correct interpretation of Eqs. (8.7), (8.8) and (8.9) would seem to be that these relations, derived earlier on quite independent grounds, ensure that the corresponding particular quantities must also be continuous at $r = 0$. But this restricted conclusion does not warrant the indiscriminate extension of Eqs. (8.5) and (8.6) to other cases without further justification.

Furthermore, there are a number of boundary conditions listed in Table 7.1 which, while not necessarily incompatible with Eqs. (8.5) and (8.6), are entirely unrelated to the latter. These are the following

For $n = 1$

$$H_1(o) - i\, G_1(o) = 0 \tag{8.10}$$

For $n = 2, 4, 6, \ldots$

$$2DH_n(o) - i\, n\, DG_n(o) = 0 \tag{8.11}$$

<u>For $n = 1, 3, 5, \ldots$</u>

$$\begin{aligned} G_n(o) &= 0 \\ H_n(o) &= 0 \end{aligned} \qquad (8.12)$$

If we proposed for some reason to derive the boundary conditions exclusively from symmetry relations and continuity assumptions, it would be necessary to abandon the conditions listed in Eqs. (8.10), (8.11) and (8.12), and to substitute for them other relations drawn from Eqs. (8.5) (8.6). At the present time no adequate physical or mathematical justification can be offered for such a change.

In an effort to clarify some of the conceptual complications associated with Eqs. (8.5) and (8.6), Johnston in Ref. (8) investigated the constraints that would theoretically have to be satisfied in order to permit the direct substitution $r = 0$ into the vorticity transport equations. His principal result pertains to the case $n \geq 2$. He found for this case that functions G_n and H_n and their first four derivatives would <u>all</u> have to vanish at $r = 0$ to allow this. His result involves ten constraints in all. Unfortunately, with only three degrees of freedom available at the axis, only three of these ten constraints can actually be enforced. While the details are different for $n = o$ and $n = 1$, the number of constraints that would theoretically be required for this purpose always exceeds the three available degrees of freedom. Hence we conclude that the simple substitution $r = 0$ into the vorticity transport equations, in their original analytical form, is never permissible. In other words, the axis remains irreducibly a singular point, and Johnston's analysis, while correct in itself, does not provide the three and only three boundary conditions needed at that point.

In view of the various considerations discussed above, it is believed that Table 7.1 provides the best overall formulation of the boundary conditions that is available at this time.

9. Status of Pipe Flow Stability Calculations

Harrison, Ref. (6), correctly formulated the basic equations for pipe flow and the boundary conditions for $n = 0$ and $n = 1$. Unfortunately, he was unable to compute results for these cases owing to a minor error in his computer program. Johnston, Ref. (8), corrected this error and computed valid results for $n = 0$, but the scope of his calculations was very limited. Johnston, in turn, formulated the boundary conditions incorrectly for all cases other than $n = 0$. Hence a specific aim of the proposed research is to amend and extend the above calculations, using the corrected boundary conditions of Table 7.1. It is felt that the revised boundary conditions for $n = 1$ are particularly likely to disclose instabilities that have not appeared before.

Despite the error in boundary conditions involved for some of the cases, Johnston's results in Ref. (8) suffice to indicate the strong probability that small negative values of α_R are definitely destabilizing. An important aim of the proposed research is to explore this question more fully and adequately. An even more exciting possibility is that <u>with the revised boundary conditions discussed above for $n = 1$, instability might possibly be found even for $\alpha_R = 0$. If so, this result would represent a very significant achievement</u>.

10. Summary of Research Objectives

The first general objective of the research outlined in this report is to carry out the calculations described at least to the extent necessary to provide significant support for the overall validity of the present theory and for the various innovations that it embodies. For the case of plane flow this entails exploring some typical portion of the boundary of incipient instability as expressed symbolically in the form

$$R_{eI}{}^{*} = R_{eI}{}^{*} \, [A, \phi, \Theta] \tag{10.1}$$

For the case of pipe flow, it entails a corresponding exploration of the form

$$R_{eI} = R_{eI} \, [\alpha_R, \alpha_I, n] \tag{10.2}$$

A particular objective of the work on pipe flow is to confirm the adequacy of the revised boundary conditions at the axis as summarized in Table 7.1. The case $n = 1$ warrants special attention as having possibly the greatest potential for disclosing instabilities not detected previously.

The ultimate objective is to carry the above calculations forward to the extent that it becomes possible to plot the two important functions symbolized below for plane flow and pipe flow, respectively, namely,

$$R_{e_{cr}} = R_{e_{cr}} (\lambda_R) \qquad \text{plane flow} \tag{10.3}$$

$$R_{e_{cr}} = R_{e_{cr}} (\alpha_R) \qquad \text{pipe flow} \tag{10.4}$$

The demonstration of the very existance of these two functions would represent a noteworthy and fundamental new contribution to the theory of hydrodynamic stability.

An essential collateral objective at all stages of the research is to report progress periodically in the technical literature. Enough progress has already been achieved to warrant at least one initial technical paper of this kind. Support is needed as much for preparing such a paper on progress already made as for continuing further analysis and calculations.

11. References

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